The Asymptotic Representation of the Elliptic Cylinder Functions.

By William Marshall.

Introduction.

We are indebted to E. Heine* for the first serious treatment of the functions of the elliptic cylinder. Starting from the defining equation

$$\frac{d^2E}{d\phi^2} + \left(\frac{8}{b}\cos 2\phi + 4z\right)E = 0,$$

where b and z are constants, Heine shows that $E(\phi)$ can be expressed in the form of the following infinite series:

$$E(\phi) = \tfrac{1}{2}a_0 + a_1 \cos 2\phi + a_2 \cos 4\phi + \ldots,$$

where the coefficients, functions of b and z, are subject to a certain recurrence formula, namely,

$$a_{n+1} = b(n^2 - z)a_n - a_{n-1}.$$

But this series, as was pointed out by Heine, is not convergent for all values of b and z, but only for certain *particular* values which he found as the roots of a certain function of b and z. This function, however, was not carefully defined by Heine, nor was his proof of the convergence of the series sufficiently rigorous to satisfy modern mathematical requirements.

S. Dannacher† cleared up the inaccuracies and supplied the deficiencies in Heine's presentation. He subjected the function whose roots determine the convergence of the series to a careful investigation, and showed how these roots might be calculated. He gave also a satisfactory proof of the convergence of the series.

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* E. Heine, Kugelfunktionen, Bd. I, § 104.

† Inaugural Dissertation, Zürich, 1906.

W. H. Butts* made a particular study of the roots of this so-called *limiting function*, and calculated for different values of b a considerable number numerically.

In the present paper, following an idea originally due to Stokes,† we obtain asymptotic or semi-convergent developments for the elliptic cylinder functions. These asymptotic expansions, which hold approximately for reasonably large values of the argument, have two decided advantages. In the first place these series are extremely well adapted to the calculation of the numerical values of the function when the argument is large, or even reasonably large; and in the second place they indicate without serious calculation the behavior of the function for large values of the argument, and particularly for an infinite argument, and they afford the most satisfactory method of determining where the function vanishes.

I. *The Transformation of the Equation.*

The equation of the functions of the elliptic cylinder may be written in the form

$$\frac{d^2U}{du^2} + (k^2\cosh^2 u + \mathfrak{B})\,U = 0, \tag{1}$$

where k and $\mathfrak{B}$ are constants. If we change the independent variable by putting $e^u = z$,‡ since

$$\cosh u = \frac{e^u + e^{-u}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

the equation (1) goes over into

$$\frac{d^2U}{dz^2} + \frac{1}{z}\cdot\frac{dU}{dz} + \left[\frac{k^2}{4}\left(1 + \frac{2}{z^2} + \frac{1}{z^4}\right) + \frac{\mathfrak{B}}{z^2}\right]U = 0. \tag{2}$$

We can remove the term containing the first derivative by putting $U_1 = Uz^{\frac{1}{2}}$. Then (2) assumes the form

$$\frac{d^2U_1}{dz^2} + \left[\frac{k^2}{4} + \left(\frac{k^2}{2} + \frac{1}{4} + \mathfrak{B}\right)\frac{1}{z^2} + \frac{k^2}{4z^4}\right]U_1 = 0. \tag{3}$$

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* Inaugural Dissertation, Zürich, 1908.

† Trans. Cambridge Phil. Soc., Vol. IX, Part I, or Math. and Phys. Papers of G. G. Stokes, Vol. II, p. 329.

‡ As a result of this transformation, or, in general, $e^{au} = z$, the singular points of the resulting equation lie only at 0 and ∞. Of this family of transformations, $a = 1$ is the only one which reduces the equation to a form such that, for large values of z, it becomes approximately (9); and this form is desirable for our present purpose.

Finally, if we put $\frac{kz}{2}=x$, (3) becomes

$$\frac{d^2U_1}{dx^2}+\left[1+\left(\frac{k^2}{2}+\frac{1}{4}+\mathfrak{B}\right)\frac{1}{x^2}+\frac{k^4}{16}\cdot\frac{1}{x^4}\right]U_1=0, \tag{4}$$

or, as we may write,

$$\frac{d^2U_1}{dx^2}+\left(1+\frac{p}{x^2}+\frac{q}{x^4}\right)U_1=0, \tag{5}$$

where the following relations exist between the quantities involved in (5) and in (1):

$$\left.\begin{aligned} p &= \frac{k^2}{2}+\frac{1}{4}+\mathfrak{B}, \\ q &= \frac{k^4}{16}, \\ U_1 &= Ue^{\frac{u}{2}}, \\ x &= \frac{ke^u}{2}. \end{aligned}\right\} \tag{6}$$

II. *The Solutions in the Neighborhood of the Singularities.*

The point $x=0$ is an essential singular point of the equation (5), as is also the point $x=\infty$. For if we put $x=\frac{1}{\xi}$, (5) becomes

$$\frac{d^2U_1}{d\xi^2}+\frac{2}{\xi}\cdot\frac{dU_1}{d\xi}+\left(\frac{1}{\xi^4}+\frac{p}{\xi^2}+q\right)U_1=0, \tag{7}$$

so that $\xi=0$ is an essential singularity of (7).

Following the known method, we might now obtain an asymptotic solution of (5) in the neighborhood of $x=0$ by putting

$$U_1=e^{\frac{\lambda}{x}}x^{\alpha}(a_0+a_1x+a_2x^2+\dots), \tag{8}$$

and then determining α, λ and the coefficients $a_0, a_1, a_2, \dots$ so that (5) is formally satisfied; it will, however, in this case, be somewhat simpler first to obtain a solution in the neighborhood of $x=\infty$, and from this to obtain the solution which holds in the neighborhood of $x=0$.

For large values of x, (5) becomes approximately

$$\frac{d^2 U_1}{dx^2} + U_1 = 0, \tag{9}$$

of which the complete solution is

$$U_1 = C_1 \cos x + C_2 \sin x. \tag{10}$$

Our solution of (5), then, which is to hold in the neighborhood of $x = \infty$ must be such that for large values of x it assumes approximately the form (10). We assume, then, as an asymptotic solution in the neighborhood of $x = \infty$,

$$U_1 = \sin x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \dots\right) + \cos x\left(B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \dots\right), \tag{11}$$

where the $A_0, A_1, A_2, \dots, B_0, B_1, B_2, \dots$ are constants which must be determined so that the equation is formally satisfied. We have, then, from (11):

$$\begin{aligned} U_1' = {} & \sin x\left(-\frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \dots\right) + \cos x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \dots\right) \\ & + \cos x\left(-\frac{B_1}{x^2} - \frac{2B_2}{x^3} - \frac{3B_3}{x^4} - \dots\right) - \sin x\left(B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \dots\right), \end{aligned}$$

$$\begin{aligned} U_1'' = {} & \sin x\left(\frac{1.2A_1}{x^3} + \frac{2.3A_2}{x^4} + \frac{3.4A_3}{x^5} + \dots\right) + 2\cos x\left(-\frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \dots\right) \\ & - \sin x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \dots\right) + \cos x\left(\frac{1.2B_1}{x^3} + \frac{2.3B_2}{x^4} + \frac{3.4B_3}{x^5} + \dots\right) \\ & - 2\sin x\left(-\frac{B_1}{x^2} - \frac{2B_2}{x^3} - \frac{3B_3}{x^4} - \dots\right) - \cos x\left(B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \dots\right). \end{aligned}$$

When we now substitute this and the value of U_1 from (11) in equation (5) and arrange according to descending powers of x, we have for the left-hand side:

$$\sin x \left\{ \begin{array}{llllll} & \frac{1.2A_1}{x^3} & + \frac{2.3A_2}{x^4} & + \frac{3.4A_3}{x^5} & + \frac{4.5A_4}{x^6} & + \dots \\ + \frac{2.1B_1}{x^2} & + \frac{2.2B_2}{x^3} & + \frac{2.3B_3}{x^4} & + \frac{2.4B_4}{x^5} & + \frac{2.5B_5}{x^6} & + \dots \\ + \frac{pA_0}{x^2} & + \frac{pA_1}{x^3} & + \frac{pA_2}{x^4} & + \frac{pA_3}{x^5} & + \frac{pA_4}{x^6} & + \dots \\ & & + \frac{qA_0}{x^4} & + \frac{qA_1}{x^5} & + \frac{qA_2}{x^6} & + \dots \end{array} \right.$$

$$
+\cos x\left\{\begin{array}{l}
\qquad\qquad\qquad \dfrac{1.2B_1}{x^3}+\dfrac{2.3B_2}{x^4}+\dfrac{3.4B_3}{x^5}+\dfrac{4.5B_4}{x^6}+\dots\\
-\dfrac{2.1A_1}{x^2}-\dfrac{2.2A_2}{x^3}-\dfrac{2.3A_3}{x^4}-\dfrac{2.4A_4}{x^5}-\dfrac{2.5A_5}{x^6}-\dots\\
+\dfrac{pB_0}{x^2}+\dfrac{pB_1}{x^3}+\dfrac{pB_2}{x^4}+\dfrac{pB_3}{x^5}+\dfrac{pB_4}{x^6}+\dots\\
\qquad\qquad\qquad +\dfrac{qB_0}{x^4}+\dfrac{qB_1}{x^5}+\dfrac{qB_2}{x^6}+\dots
\end{array}\right.
$$

In order that this vanish, the coefficients of $\cos x$ and $\sin x$ must vanish separately. The necessary conditions for this are that the coefficients of the various powers of x, namely, x^{-2}, x^{-3}, x^{-4}, ..., should vanish separately. This gives us the following relations for determining the A's and B's in terms of any two of them, say in terms of A_0 and B_0:

$$
\left.\begin{array}{l}
A_1=\dfrac{p}{2}B_0=B_0f_1(p,q)\ \text{(say)},\\
B_1=-\dfrac{p}{2}A_0=-A_0f_1,\\
A_2=\dfrac{B_1(p+2)}{2.2}=-\dfrac{A_0}{2^2.2}p(p+2)=-A_0f_2,\\
B_2=-\dfrac{A_1(p+2)}{2.2}=-\dfrac{B_0}{2^2.2}p(p+2)=-B_0f_2,\\
A_3=B_0\left[\dfrac{-p(p+2)(p+6)}{2^3.3!}+\dfrac{q}{2.3}\right]=-B_0f_3,\\
B_3=A_0\left[\dfrac{p(p+2)(p+6)}{2^3.3!}-\dfrac{q}{2.3}\right]=A_0f_3,\\
A_4=A_0\left[\dfrac{p(p+2)(p+6)(p+12)}{2^4.4!}-\dfrac{q(p+12)}{2.4.2.3}-\dfrac{pq}{2.1.2.4}\right]=A_0f_4,\\
B_4=B_0\left[\dfrac{p(p+2)(p+6)(p+12)}{2^4.4!}-\dfrac{q(p+12)}{2.4.2.3}-\dfrac{pq}{2.1.2.4}\right]=B_0f_4,\\
A_5=B_0\left[\dfrac{p(p+2)(p+6)(p+12)(p+20)}{2^5.5!}-\dfrac{(p+12)(p+20)}{2.4.2.3.2.5}\right.\\
\qquad\qquad\left.-\dfrac{pq(p+20)}{2.1.2.4.2.5}-\dfrac{pq(p+2)}{2^5.2^2.2}\right]=B_0f_5,\\
B_5=-A_0f_5,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}\right\}\quad(12)
$$

Here $f_1, f_2, f_3, \ldots.$ are simply abbreviations. We have, in general,

$$\left.\begin{aligned} A_n &= \frac{B_{n-1}[p+n(n-1)]}{2(n-1)} + \frac{B_{n-3}q}{2(n-1)}, \\ B_n &= -\frac{A_{n-1}[p+n(n-1)]}{2(n-1)} - \frac{A_{n-3}q}{2(n-1)}. \end{aligned}\right\} \tag{13}$$

From these relations we may determine the constants $A_1, A_2, A_3, \ldots.,$ $B_1, B_2, B_3, \ldots.$ as far as we choose, though, as might have been expected, it is not possible to give general formulas for A_n and B_n except in terms of the coefficients which immediately precede them. Substituting now in (11) the values of the coefficients as given by (12), we thus obtain for the asymptotic solution of (5) for large values of x the following:

$$\begin{aligned} U_1 = \sin x\left[A_0 + \frac{B_0 f_1}{x} - \frac{A_0 f_2}{x^2} - \frac{B_0 f_3}{x^3} + \frac{A_0 f_4}{x^4} + \ldots.\right] \\ + \cos x\left[B_0 - \frac{A_0 f_1}{x} - \frac{B_0 f_2}{x^2} + \frac{A_0 f_3}{x^3} + \frac{B_0 f_4}{x^4} - \ldots.\right]. \end{aligned} \tag{14}$$

This we may write in the form

$$\begin{aligned} U_1 = A_0\left(\sin x - \frac{f_1 \cos x}{x} - \frac{f_2 \sin x}{x^2} + \frac{f_3 \cos x}{x^3} + \ldots.\right) \\ + B_0\left(\cos x + \frac{f_1 \sin x}{x} - \frac{f_2 \cos x}{x^2} - \frac{f_3 \sin x}{x^3} + \ldots.\right). \end{aligned} \tag{15}$$

If now we change the arbitrary constants by putting

$$\left.\begin{aligned} A_0 \sin x + B_0 \cos x &= C \cos(\alpha - x), \\ -(A_0 \cos x - B_0 \sin x) &= C \sin(\alpha - x), \end{aligned}\right\} \tag{16*}$$

we may write (14) or (15) in the form

$$\begin{aligned} U_1 = C\cos(\alpha - x)\left[1 - \frac{f_2}{x^2} + \frac{f_4}{x^4} - \frac{f_6}{x^6} + \ldots.\right] \\ + C\sin(\alpha - x)\left[\frac{f_1}{x} - \frac{f_3}{x^3} + \frac{f_5}{x^5} - \frac{f_7}{x^7} + \ldots.\right]; \end{aligned} \tag{17}$$

or we may write

$$U_1 = C[P\cos(\alpha - x) + Q\sin(\alpha - x)], \tag{18}$$

* This amounts to putting

$$C = \sqrt{A_0^2 + B_0^2},$$
$$\alpha = \tan^{-1}\frac{A_0}{B_0}.$$

where P and Q denote the following infinite series:

$$\left.\begin{aligned} P &= 1 - \frac{f_2}{x^2} + \frac{f_4}{x^4} - \frac{f_6}{x^6} + \cdots, \\ Q &= \frac{f_1}{x} - \frac{f_3}{x^3} + \frac{f_5}{x^5} - \frac{f_7}{x^7} + \cdots. \end{aligned}\right\} \tag{19}$$

If we return to the original variables U and u, we may write (18) in the form (see the relations (6))

$$U = \frac{C}{e^{\frac{u}{2}}}\left[P\cos\left(\alpha - \frac{ke^u}{2}\right) + Q\sin\left(\alpha - \frac{ke^u}{2}\right)\right], \tag{20}$$

where, in P and Q, x is replaced by $\frac{ke^u}{2}$ and p and q by their equivalents.

The series denoted by P and Q are at first, for reasonably large values of x, rapidly convergent; they are both, however, ultimately rapidly divergent, as may be seen from the relations (12). The point at which they begin to diverge depends on the relative magnitude of f_n and x^n. In fact if we stop after taking $2n$ terms of the solution (14) (n terms containing $\sin x$ and n containing $\cos x$) and set the result in (5), we obtain

$$\frac{d^2U_1}{dx^2} + \left(1 + \frac{p}{x^2} + \frac{q}{x^4}\right)U_1 = \frac{f_{n+1}}{x^{n+1}}\left[A_0\sin\left(x + \frac{n-1}{2}\pi\right) + B_0\cos\left(x + \frac{n-1}{2}\pi\right)\right]; \tag{21}$$

and the right-hand side of this is small if $\frac{f_{n+1}}{x^{n+1}}$ is also small. In using such series in numerical computation, we need not take all the terms up to and including the smallest; the number taken will depend upon the closeness of the approximation desired.

To obtain a solution of equation (5) which holds in the neighborhood of the other singularity $x = 0$, we proceed most conveniently as follows: in (5) we put $x = \frac{1}{t}$; (5) becomes then

$$\frac{d^2U_1}{dt^2} + \frac{2}{t}\frac{dU_1}{dt} + \left(\frac{1}{t^4} + \frac{p}{t^2} + q\right)U_1 = 0. \tag{22}$$

We can get rid of the term containing the first derivative by putting $U_1 = U_2t^{-1}$. Then (22) becomes

$$U_2'' + \left(\frac{1}{t^4} + \frac{p}{t^2} + q\right)U_2 = 0. \tag{23}$$

Finally, if we put $t = \xi q^{-\frac{1}{2}}$, we obtain

$$\frac{d^2 U_2}{d\xi^2} + \left(1 + \frac{p}{\xi^2} + \frac{q}{\xi^4}\right) U_2 = 0. \tag{24}$$

Equation (24) is, however, of exactly the same form as (5), so that we may use the former solution (14) or (15). The solution of (24) which is valid in the neighborhood of $\xi = \infty$ is then

$$\begin{aligned} U_2 = \sin \xi \Big(a_0 + \frac{b_0 f_1}{\xi} - \frac{a_0 f_2}{\xi^2} - \frac{b_0 f_3}{\xi^3} + \ldots \\ + \cos \xi \Big(b_0 - \frac{a_0 f_1}{\xi} - \frac{b_0 f_2}{\xi^2} + \frac{a_0 f_3}{\xi^3} + \ldots, \end{aligned} \tag{25}$$

where the $f_1, f_2, f_3, \ldots$ have exactly the same meaning as in (12). If now in (25) we restore the values of the variables U_2 and ξ, we have

$$\begin{aligned} U_1 = x \sin \frac{q^{\frac{1}{2}}}{x} \Big(a_0 + \frac{x\, b_0 f_1}{q^{\frac{1}{2}}} - \frac{x^2 a_0 f_2}{q} - \frac{x^3 b_0 f_3}{q^{\frac{3}{2}}} + \ldots \\ + x \cos \frac{q^{\frac{1}{2}}}{x} \Big(b_0 - \frac{x\, a_0 f_1}{q^{\frac{1}{2}}} - \frac{x^2 b_0 f_2}{q} + \frac{x^3 a_0 f_3}{q^{\frac{3}{2}}} + \ldots \end{aligned} \tag{26}$$

We may write (26) in the form

$$\begin{aligned} U_1 = a_0 x \Big\{ &\sin \frac{q^{\frac{1}{2}}}{x} \left(1 - \frac{x^2}{q} f_2 + \frac{x^4}{q^2} f_4 - \ldots\right) \\ &+ \cos \frac{q^{\frac{1}{2}}}{x} \left(- \frac{x}{q^{\frac{1}{2}}} f_1 + \frac{x^3}{q^{\frac{3}{2}}} f_3 - \frac{x^5}{q^{\frac{5}{2}}} f_5 + \ldots\right)\Big\} \\ + b_0 x \Big\{ &\sin \frac{q^{\frac{1}{2}}}{x} \left(\frac{x}{q^{\frac{1}{2}}} f_1 - \frac{x^3}{q^{\frac{3}{2}}} f_3 + \frac{x^5}{q^{\frac{5}{2}}} f_5 - \ldots\right) \\ &+ \cos \frac{q^{\frac{1}{2}}}{x} \left(1 - \frac{x^2}{q} f_2 + \frac{x^4}{q^2} f_4 - \ldots\right)\Big\}. \end{aligned} \tag{27}$$

We may adopt then, as the two fundamental integrals of equation (5) which are valid asymptotically in the neighborhood of $x = 0$:

$$U_{1p} = x\left(A \sin \frac{q^{\frac{1}{2}}}{x} - B \cos \frac{q^{\frac{1}{2}}}{x}\right), \tag{28}$$

$$U_{1q} = x\left(B \sin \frac{q^{\frac{1}{2}}}{x} + A \cos \frac{q^{\frac{1}{2}}}{x}\right), \tag{29}$$

where

$$\left.\begin{aligned} A &= 1 - \frac{x^2}{q} f_2 + \frac{x^4}{q^2} f_4 - \frac{x^6}{q^3} f_6 + \ldots, \\ B &= \phantom{1 -{}} \frac{x}{q^{\frac{1}{2}}} f_1 - \frac{x^3}{q^{\frac{3}{2}}} f_3 + \frac{x^5}{q^{\frac{5}{2}}} f_5 - \ldots \end{aligned}\right\} \tag{30}$$

These integrals have no meaning when $q = 0$. This case, however, we need not consider, since for $q = 0$ (5) becomes

$$U_1'' + \left(1 + \frac{p}{x^2}\right)U_1 = 0, \tag{31}$$

and this is a reduced Bessel equation with $n^2 = \frac{1}{4} - p$.*

It remains to bring the integrals (18) and (28) or (29) into connection; that is, to determine the constants C and α in (18) so that (18) shall be an approximation to the fundamental integrals which are denoted by U_{1p} and U_{1q} in (28) and (29). In general this is brought about by expressing the function, here U_{1p} or U_{1q}, in the form of a definite integral and then, from this, determining the leading term in the function as the argument increases. But the expressing of these functions U_{1p} and U_{1q} in the form of definite integrals of sufficient simplicity seems to involve difficulties not as yet overcome.† However, for any particular case, for any particular pair of values of p and q, the constants C and α might be determind in the following manner: Having chosen a value of x for which (18) and (28), for example, give a sufficiently close approximation, compute the numerical values of these expressions (18) and (28) for this value of x. This would give one relation between C and α. By proceeding in a similar way for a second value of x, the values of C and α would be determined.‡

III. *The Roots of* $U_\alpha = 0$.

Although we have not determined the constants C and α in (18) so that (18) shall be an approximation to the fundamental integrals U_{1p} and U_{1q}, yet it is possible to determine from (18) the general behavior of these functions for large values of x. For the sake of simplicity we carry the discussion through with the variables U_1 and x and denote by $U_{1\alpha}$ either of the integrals U_{1p} or U_{1q}, where the α will serve to indicate that on the value of the constants α and C

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* That is, if in (31) we put $U_1 = Jx^{\frac{1}{2}}$, we have, after reduction,

$$J'' + \frac{J'}{x} + J\left(1 + \frac{p - \frac{1}{4}}{x^2}\right) = 0;$$

and this is the ordinary form of the Bessel equation with n^2 replaced by $\frac{1}{4} - p$.

† See III, where the value of α is approximately determined, but not in this general manner.

‡ See the numerical example in V. Here the arbitrary constant is determined in the above-mentioned way.

will depend whether the one or the other of these integrals is intended. We have then, from (18),

$$U_{1a} = C[P\cos(\alpha - x) + Q\sin(\alpha - x)]; \tag{31}$$

and we proceed to find where U_{1a} vanishes. We put

$$P = M\cos\psi,\quad Q = M\sin\psi, \tag{32}$$

where, as before,

$$\left.\begin{aligned} P &= 1 - \frac{f_2}{x^2} + \frac{f_4}{x^4} - \frac{f_6}{x^6} + \ldots, \\ Q &= \frac{f_1}{x} - \frac{f_3}{x^3} + \frac{f_5}{x^5} - \frac{f_7}{x^7} + \ldots \end{aligned}\right\} \tag{19}$$

(31) becomes then

$$U_{1a} = CM[\cos(\alpha - x)\cos\psi + \sin(\alpha - x)\sin\psi], \tag{33}$$

$$U_{1a} = CM\cos(\psi - \alpha + x), \tag{34}$$

where, from (32), since

$$\left.\begin{aligned} M &= \sqrt{P^2 + Q^2}, \\ \psi &= \tan^{-1}\frac{Q}{P}, \end{aligned}\right\} \tag{35}$$

$$\psi = \tan^{-1}\left(\frac{f_1}{x} + \frac{f_1 f_2 - f_3}{x^3} + \frac{f_5 - f_1 f_4 - f_1 f_2^2 + f_2 f_3}{x^5} + \ldots\right), \tag{36}$$

$$M = 1 - \frac{f_2 - \frac{f_1^2}{2}}{x^2} + \frac{f_4 - f_1 f_3 + \frac{f_1^2 f_2}{2} - \frac{f_1^4}{8}}{x^4} - \ldots \tag{37}$$

Now, from (34), U_{1a} will be zero when

$$\cos(\psi - \alpha + x) = 0;$$

that is, when

$$\psi - \alpha + x = \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots,\ \frac{2n-1}{2}.\pi;$$

that is, when

$$x = \alpha + \frac{2n-1}{2}.\pi - \psi \tag{38}$$

(where n is an integer). This equation we have to solve for x on the supposition that x is large. If in (38) we put, for shortness, $\alpha + \frac{2n-1}{2}.\pi = \phi$, then we have to solve

$$x = \phi - \psi. \tag{39}$$

If now we expand ψ in terms of x by means of the expansion of $\tan^{-1} z$, namely

$$\tan^{-1} z = z - \frac{1}{3} z^3 + \frac{1}{5} z^5 - \frac{1}{7} z^7 + \dots,$$

we obtain, after reduction,

$$\psi = \frac{f_1}{x} + \frac{f_1 f_2 - f_3 - \frac{f_1^3}{3}}{x^3} + \frac{f_5 - f_1 f_4 - f_1 f_2^2 + f_2 f_3 - f_1^3 f_2 + f_1^2 f_2 + \frac{f_1^5}{5}}{x^5} + \dots \tag{40}$$

Putting this value of ψ in (39) and solving the resulting equation, namely

$$\begin{aligned} x = \phi - \frac{f_1}{x} - \frac{f_1 f_2 - f_3 - \frac{f_1^3}{3}}{x^3} \\ - \frac{f_5 - f_1 f_4 - f_1 f_2^2 + f_2 f_3 - f_1^3 f_2 + f_1^2 f_3 + \frac{f_1^5}{5}}{x^5} + \dots, \end{aligned} \tag{41}$$

by the method of successive approximations, we obtain

$$\begin{aligned} x = \phi - f_1 \cdot \frac{1}{\phi} + \left(f_3 - f_1 f_2 + \frac{f_1^3}{3} - f_1^2\right) \cdot \frac{1}{\phi^3} \\ - \Big(f_5 - f_1 f_4 - f_2 f_3 + f_1 f_2^2 + f_1^2 f_3 - f_1^3 + \frac{f_1^5}{5} \\ + 4 f_1 f_3 - 4 f_1^2 f_2 + \frac{f_1^4}{3} + 2 f_1^3\Big) \frac{1}{\phi^5} + \dots \end{aligned} \tag{42}$$

We have, then, for the general formula for the nth root of the equation $U_{1\alpha} = 0$, the following:

$$x = \alpha + \frac{2n-1}{2}\pi - \frac{f_1}{\alpha + \frac{2n-1}{2}\pi} + \frac{f_3 - f_1 f_2 + \frac{f_1^3}{3} + f_1^2}{\left(\alpha + \frac{2n-1}{2}\pi\right)^3} - \dots \tag{43}$$

By means of the relations (12) we may express this in terms of p and q. We find, after some reduction,

$$x = \alpha + \frac{2n-1}{2}\pi - \frac{p}{2\alpha + (2n-1)\pi} + \frac{5p^2 + 6p - 12q}{3[2\alpha + (2n-1)\pi]^3} - \dots \tag{44}$$

By means of the relations (6) this may be transformed into terms of the original quantities U, u, k and $\mathfrak{B}$.

The expression (44) shows clearly how the roots are spaced along the axis. It shows also distinctly the influence of the undetermined constant on the position of the roots. The leading term in (44) is of course $\alpha + \frac{2n-1}{2}\pi$, so that a change in the value of α amounts to a displacement of the roots along the axis. (44) shows also how the position of the roots depends upon the values of p and q. In (44) q appears first in the fourth term, which for ordinary values of p and q, on account of the rapid convergence of the series, will be small, so that a change in the value of q will produce a very slight change in the position of the roots.

In the calculations involved in the preceding pages — that is, in finding the values of ψ as given in (36) and (40), and M as given in (37), and in solving the equation (41) — we may proceed as there indicated, performing the arithmetical operations upon the infinite series there involved, and carrying the results to the desired degree of accuracy. These results may be obtained more briefly (particularly if many terms are desired) and much more elegantly by the following method.*

We had found, as a solution of (5),

$$U_{1\alpha} = C[P\cos(\alpha - x) + Q\sin(\alpha - x)]. \tag{31}$$

If now we put

$$P = M\cos\psi, \quad Q = M\sin\psi, \tag{32}$$

and further set $C = 1$, $\alpha = 0$, we have, as a particular solution of (5),

$$U_1 = M\cos(\psi + x). \tag{45}$$

By substituting this integral in the differential equation (5), and determining the coefficients so that (5) is satisfied, we should be able to deduce recurrence formulas for the coefficients of M and ψ. We have then, from (45),

$$U_1' = M'\cos(x+\psi) - M\sin(x+\psi)(1+\psi'), \tag{46}$$

$$\begin{aligned} U_1'' = M''\cos(x+\psi) &- 2M'\sin(x+\psi)(1+\psi') \\ &- M\sin(x+\psi)\psi'' - M\cos(x+\psi)(1+\psi')^2. \end{aligned} \tag{47}$$

* This method was suggested to me by Professor Burkhardt, who, however, has not published anything on the subject. It is generally applicable in similar problems, being particularly advantageous in the calculation of the roots of Bessel functions.

When we substitute these results in (5), we have

$$M'' \cos(x+\psi) - 2M' \sin(x+\psi)(1+\psi') - M \sin(x+\psi)\,\psi''$$
$$- M\cos(x+\psi)(1+\psi')^2 + \left(1+\frac{p}{x^2}+\frac{q}{x^4}\right) M\cos(x+\psi) = 0. \tag{48}$$

In this the coefficients of $\sin(x+\psi)$ and $\cos(x+\psi)$ must vanish separately; this gives the two equations for determining M and ψ:

$$M'' - M(1+\psi')^2 + \left(1+\frac{p}{x^2}+\frac{q}{x^4}\right) M = 0, \tag{49}$$

$$2M'(1+\psi') + M\psi'' = 0. \tag{50}$$

Equation (50) is readily integrated, giving

$$1+\psi' = \frac{c_1}{M^2}. \tag{51}$$

When we substitute this value of $1+\psi'$ in (49), we have a differential equation for M, namely

$$M'' - \frac{c_1}{M^3} + \left(1+\frac{p}{x^2}+\frac{q}{x^4}\right) M = 0. \tag{52}$$

Now, since we are concerned only with a particular value of M, we may assign to the constant of integration c_1 any value we choose. As we shall see later, it will be most advantageous to take $c_1 = 1$. We know from the equations (19) and from the defining equation for M, namely $M = \sqrt{P^2+Q^2}$, that M must be an even function of x, and moreover that the first term is 1. We therefore assume

$$M = 1 + \frac{k_2}{x^2} + \frac{k_4}{x^4} + \frac{k_6}{x^6} + \ldots. \tag{53}$$

Then follow

$$M' = -\frac{2k_2}{x^3} - \frac{4k_4}{x^5} - \frac{6k_6}{x^7} + \ldots, \tag{54}$$

$$M'' = \frac{3\,.\,2k_2}{x^4} + \frac{5\,.\,4k_4}{x^6} + \frac{7\,.\,6k_6}{x^8} + \ldots. \tag{55}$$

Also, by the binomial theorem,

$$\frac{1}{M^3} = 1 + \frac{a_2}{x^2} + \frac{a_4}{x^4} + \frac{a_6}{x^6} + \ldots., \tag{56}$$

where the $a_2, a_4, a_6, \ldots.$ are given by the relations:

$$\left.\begin{aligned} a_2 &= -3k_2, \\ a_4 &= -\frac{4k_2a_2}{2} - 3k_4, \\ a_6 &= \frac{-5k_2a_4 - 7k_4a_2}{3} - 3k_6, \\ a_8 &= \frac{-6k_2a_6 - 8k_4a_4 - 10k_6a_2}{4} - 3k_8, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{57}$$

If now we put the value of M'' from (55), M from (51), and $\frac{1}{M^3}$ from (56) in (52), we have, after putting $c_1 = 1$,

$$\left.\begin{aligned} & \qquad\quad \frac{3\,.\,2\,k_2}{x^4} + \frac{5\,.\,4\,k_4}{x^6} + \frac{7\,.\,6\,k_6}{x^8} + \ldots. \\ & + \frac{k_2}{x^2} + \frac{k_4}{x^4} + \frac{k_6}{x^6} + \frac{k_8}{x^8} + \ldots. \\ & + \frac{p}{x^2} + \frac{pk_2}{x^4} + \frac{pk_4}{x^6} + \frac{pk_6}{x^8} + \ldots. \\ & \qquad\quad + \frac{q}{x^4} + \frac{qk_2}{x^6} + \frac{qk_4}{x^8} + \ldots. \\ & - \frac{a_2}{x^2} - \frac{a_4}{x^4} - \frac{a_6}{x^6} - \frac{a_8}{x^8} - \ldots. = 0. \end{aligned}\right\} \tag{58}$$

When the coefficients of the various powers of x are put separately equal to zero, the following recurrence formulas are obtained:

$$\left.\begin{aligned} 4k_2 &= -p, \\ 4k_4 &= -(p+6)\,k_2 - q + b_4, \\ 4k_6 &= -(p+20)\,k_4 - qk_2 + b_6, \\ 4k_8 &= -(p+42)\,k_6 - qk_4 + b_8, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 4k_n &= -[p + (n-1)(n-2)]\,k_{n-2} - qk_{n-4} + b_n, \end{aligned}\right\} \tag{59}$$

where b_4, b_6, b_8, ... are the same as a_4, a_6, a_8, ... as given in equation (57), only with the last term dropped; that is,

$$\left.\begin{aligned} b_2 &= 0, \\ b_4 &= -\frac{4k_2a_2}{2}, \\ b_6 &= \frac{-5k_2a_4 - 7k_4a_2}{3}, \\ b_8 &= \frac{-6k_2a_6 - 8k_4a_4 - 10k_6a_2}{4}, \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{60}$$

We have then, in (59), a formula for any coefficient in the expansion of M expressed in terms of the preceding coefficients, so that any coefficient can be at once expressed in terms of p and q; naturally, for the later coefficients the calculation becomes somewhat laborious, owing to the complicated character of a_2, a_4, a_6, ... when expressed in terms of p and q. Below will be found a list of the first few coefficients calculated in this way.

$$\left.\begin{aligned} k_2 &= -\frac{p}{4}, \\ k_4 &= \frac{5p^2}{32} + \frac{3p}{8} - \frac{q}{4}, \\ k_6 &= -\frac{15p^3}{28} - \frac{37p^2}{32} - \frac{15p}{8} + q\left(\frac{5p}{16} + \frac{5}{4}\right), \\ k_8 &= \frac{195p^4}{2048} + \frac{611p^3}{256} + \frac{1821p^2}{128} + \frac{315p}{16} - \frac{45p^2q}{128} - \frac{157pq}{32} + \frac{5q^2}{32} - \frac{105q}{8}. \end{aligned}\right\} \tag{61}$$

We may use these coefficients of the expansion of M to good advantage in the further computation, particularly in the task of solving by the method of approximation the equation

$$x = \phi - \psi. \tag{39}$$

To show how this may be done, we have from (51), after putting $c = 1$,

$$\psi' = \frac{1}{M^2} - 1. \tag{62}$$

Expanding M^{-2} by the binomial theorem, we have

$$\psi' = \frac{c_2}{x^2} + \frac{c_4}{x^4} + \frac{c_6}{x^6} + \frac{c_8}{x^8} + \cdots, \tag{63}$$

where $c_2, c_4, c_6, \ldots.$ are given by

$$\left.\begin{aligned} c_2 &= -2k_2, \\ 2c_4 &= -3k_2c_2 - 4k_4, \\ 3c_6 &= -4k_2c_4 - 5k_4c_2 - 6k_6, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{64}$$

If we now integrate (63), putting the constant of integration equal to zero, we have

$$\psi = -\frac{c_2}{x} - \frac{c_4}{3x^3} - \frac{c_6}{5x^5} - \ldots., \tag{65}$$

from which ψ can be expressed in terms of p and q if desired. In order to solve the equation (39), however, we may use the value of ψ as given in (65). We have then to solve

$$x = \phi + \frac{c_2}{x^2} + \frac{c_4}{3x^3} + \frac{c_6}{5x^5} + \frac{c_8}{7x^7} + \ldots. \tag{66}$$

The successive approximations are readily found to be:

$$\left.\begin{aligned} x_1 &= \phi, \\ x_2 &= \phi + \frac{c_2}{\phi}, \\ x_3 &= \phi + \frac{c_2}{\phi} + \frac{\frac{c_4}{3} - c_2^2}{\phi^3}, \\ x_4 &= \phi + \frac{c_2}{\phi} + \frac{\frac{c_4}{3} - c_2^2}{\phi^3} + \frac{2c_2^3 - \frac{4c_2c_4}{3} + \frac{c_6}{5}}{\phi^5}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}\right\} \tag{67}$$

and by means of (64) and (61) these can be expressed in terms of p and q, giving the results previously obtained in (44).

The fact that in (44) the value of the roots of $U_{1a} = 0$ depends only on the undetermined constant α suggests the following method of determining approximately the value of this constant. We have

$$U_{1p} = x\left(A \sin\frac{\sqrt{q}}{x} - B \cos\frac{\sqrt{q}}{x}\right), \tag{28}$$

where A and B have the values given in (30). If we put

$$\left.\begin{aligned} A &= N\cos\theta, \\ B &= N\sin\theta, \end{aligned}\right\} \tag{68}$$

(28) becomes

$$U_{1p} = xN\left(\sin\frac{\sqrt{q}}{x}\cos\theta - \cos\frac{\sqrt{q}}{x}\sin\theta\right), \tag{69}$$

$$U_{1p} = xN\sin\left(\frac{\sqrt{q}}{x} - \theta\right). \tag{70}$$

This will vanish if

$$\sin\left(\frac{\sqrt{q}}{x} - \theta\right) = 0\,; \tag{71}$$

that is, if

$$x = \frac{\sqrt{q}}{\theta + n\pi} \qquad (n = 0, 1, 2, 3, \ldots.). \tag{72}$$

We find now, as in (40),

$$\begin{aligned} \theta = \frac{xf_1}{q^{\frac{1}{2}}} &+ \frac{x^3}{q^{\frac{3}{2}}}\left(f_1f_2 - f_3 - \frac{f_1^3}{3}\right) \\ &+ \frac{x^5}{q^{\frac{5}{2}}}\left(f_5 - f_1f_4 - f_1f_2^2 + f_2f_3 - f_1^3f_2 + f_1^2f_3 + \frac{f_1^5}{5}\right) + \ldots. \end{aligned} \tag{73}$$

We have now to solve approximately the equation

$$x = \frac{\sqrt{q}}{n\pi + \frac{xf_1}{q^{\frac{1}{2}}} + \frac{x^3}{q^{\frac{3}{2}}}\left(f_1f_2 - f_3 - \frac{f_1^3}{3}\right) + \ldots.}, \tag{74}$$

on the supposition that x is small.

It will be more convenient to write (74) in the form

$$x = \sqrt{q}\left(\frac{1}{n\pi} - \frac{xf_1}{q^{\frac{1}{2}}(n\pi)^2} + \frac{f_1^2x^2}{q(n\pi)^3} - \frac{x^3}{q^{\frac{3}{2}}}\left(\frac{f_1^3}{(n\pi)^4} + \frac{f_1f_2 - f_3 - f_1^3}{(n\pi)^2}\right) + \ldots.\right). \tag{75}$$

The successive approximations are:

$$\left.\begin{aligned} x_1 &= \frac{\sqrt{q}}{n\pi} = l_1 \text{ (say)}, \\ x_2 &= \sqrt{q}\left(\frac{1}{n\pi} - \frac{f_1}{(n\pi)^3}\right) = l_2, \\ x_3 &= \sqrt{q}\left(\frac{1}{n\pi} - \frac{f_1}{(n\pi)^3} + \frac{2f_1^2}{(n\pi)^5}\right) = l_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{76}$$

If we put this approximate value of x equal to the approximate value obtained in (44), we have an equation containing only α. This is

$$\alpha+\frac{2n-1}{2}\pi=l_n+\frac{p}{2\left(\alpha+\frac{2n-1}{2}\pi\right)}-\frac{5p^2+6p-12q}{24\left(\alpha+\frac{2n-1}{2}\pi\right)^3}+\dots \tag{77}$$

This gives, as successive approximations for α:

$$\left.\begin{array}{l}\alpha_1+\frac{2n-1}{2}\pi=l_n,\\ \alpha_2+\frac{2n-1}{2}\pi=l_n+\frac{p}{2l_n},\\ \alpha_3+\frac{2n-1}{2}\pi=l_n+\frac{p}{2l_n}-\frac{11p^2+6p-12q}{24l_n^3},\\ \dots\dots\dots\dots\dots\dots\dots\dots\end{array}\right\} \tag{78}$$

This approximate value of α, which should hold for certain values of n neither too large nor too small, would indicate that the true value of α probably is a rather complicated function of the two parameters of the equation, p and q.

In a similar way the value of α might be determined so that (44) would give approximately the position of the roots of the other fundamental integral U_{1q}.

IV. *The Roots of* $U'_\alpha=0$.

In a similar way we may compute the roots of $U'_{1\alpha}=0$; that is, we may find the maxima and minima of $U_{1\alpha}$. We have

$$U_\alpha=\frac{C}{e^{\frac{u}{2}}}\left[P\cos\left(\alpha-\frac{ke^u}{2}\right)+Q\sin\left(\alpha-\frac{ke^u}{2}\right)\right], \tag{20}$$

where

$$\left.\begin{array}{l}P=\ 1\ -\frac{2^2f_2}{k^2e^{2u}}+\frac{2^4f_4}{k^4e^{4u}}-\frac{2^6f_6}{k^6e^{6u}}+\dots,\\ Q=\frac{2f_1}{ke^u}-\frac{2^3f_3}{k^3e^{3u}}+\frac{2^5f_5}{k^5e^{5u}}-\dots\end{array}\right\} \tag{79}$$

Now, since U'_α exists, and indeed also in the form of an asymptotic expansion, we may differentiate (20) term by term, the results having of course the same meaning as the original expansion. We thus obtain

$$\begin{aligned}U'_\alpha=\frac{C}{e^{\frac{u}{2}}}\Big[\frac{k}{2}e^u\,.\,P\sin\left(\alpha-\frac{ke^u}{2}\right)+\cos\left(\alpha-\frac{ke^u}{2}\right)P'-\frac{Qke^u}{2}\cos\left(\alpha-\frac{ke^u}{2}\right)\\ +\sin\left(\alpha-\frac{ke^u}{2}\right)\,.\,Q'\Big]-\frac{1}{2e^{\frac{u}{2}}}\left[P\cos\left(\alpha-\frac{ke^u}{2}\right)+Q\sin\left(\alpha-\frac{ke^u}{2}\right)\right].\end{aligned} \tag{80}$$

If we put this equal to zero, we have

$$\sin\left(\alpha-\frac{ke^u}{2}\right)\left[\frac{ke^uP}{2}+Q'-\frac{Q}{2}\right]+\cos\left(\alpha-\frac{ke^u}{2}\right)\left[P'-\frac{ke^uQ}{2}-\frac{P}{2}\right]=0. \tag{81}$$

From (79) we have

$$\left.\begin{aligned} P' &= \frac{2^2 \cdot 2f_2}{k^2e^{2u}} - \frac{2^4 \cdot 4f_4}{k^4e^{4u}} + \frac{2^6 \cdot 6f_6}{k^6e^{6u}} + \dots, \\ Q' &= -\frac{2f_1}{ke^u} + \frac{2^3 \cdot 3f_3}{k^3e^{3u}} - \frac{2^5 \cdot 5f_5}{k^5e^{5u}} + \dots \end{aligned}\right\} \tag{82}$$

For convenience we may return to the former variable, putting, as before, $x=\frac{ke^u}{2}$, so that we have

$$\left.\begin{aligned} P &= 1-\frac{f_2}{x^2}+\frac{f_4}{x^4}-\frac{f_6}{x^6}+\dots, \\ Q &= \frac{f_1}{x}-\frac{f_3}{x^3}+\frac{f_5}{x^5}-\frac{f_7}{x^7}+\dots; \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} P' &= \frac{2f_2}{x^2}-\frac{4f_4}{x^4}+\frac{6f_6}{x^6}-\frac{8f_8}{x^8}+\dots, \\ Q' &= -\frac{f_1}{x}+\frac{3f_3}{x^3}-\frac{5f_5}{x^5}+\frac{7f_7}{x^7}-\dots \end{aligned}\right\} \tag{83}$$

Then the expressions in the square brackets in (81) become

$$\left.\begin{aligned} xP+Q'-\frac{Q}{2} &= x-\frac{f_2+\frac{3}{2}f_1}{x^3}-\frac{f_6+\frac{11}{2}f_5}{x^5}+\dots = R \text{ (say)}, \\ P'-xQ-\frac{P}{2} &= -f_1-\frac{1}{2}+\frac{f_3+\frac{5}{2}f_2}{x^2}-\frac{f_5+\frac{9}{2}f_4}{x^4}+\frac{f_7+\frac{13}{2}f_6}{x^6}-\dots = S. \end{aligned}\right\} \tag{84}$$

We put now, for shortness,

$$\left.\begin{aligned} f_1+\frac{1}{2} &= -g_1(p,q), \text{ (or simply } -g_1), \\ f_2+\frac{3}{2}f_1 &= g_2, \\ f_3+\frac{5}{2}f_2 &= -g_3, \\ f_4+\frac{7}{2}f_3 &= g_4, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{85}$$

Then the series in (84) become

$$\left.\begin{aligned} R &= x\left[1 - \frac{g_2}{x^2} + \frac{g_4}{x^4} - \frac{g_6}{x^6} + \cdots\right], \\ S &= x\left[\frac{g_1}{x} - \frac{g_3}{x^3} + \frac{g_5}{x^5} - \frac{g_7}{x^7} + \cdots\right]. \end{aligned}\right\} \tag{86}$$

The equation (81) becomes then, after dividing by x,

$$R_1 \sin(\alpha - x) + S_1 \cos(\alpha - x), \tag{87}$$

where

$$\left.\begin{aligned} R_1 &= 1 - \frac{g_2}{x^2} + \frac{g_4}{x^4} - \frac{g_6}{x^6} + \cdots, \\ S_1 &= \frac{g_1}{x} - \frac{g_3}{x^3} + \frac{g_5}{x^5} - \frac{g_7}{x^7} + \cdots \end{aligned}\right\} \tag{88}$$

As in III, we put

$$\left.\begin{aligned} R_1 &= M_1 \cos\psi_1, \\ S_1 &= M_1 \sin\psi_1, \end{aligned}\right\} \tag{89}$$

where M_1 and ψ_1 have exactly the same form as the M and ψ in (37) and (40), except that the g-functions now replace the f-functions. Then (87) becomes

$$M_1\left[\sin(\alpha - x)\cos\psi_1 + \cos(\alpha - x)\sin\psi_1\right] = 0, \tag{90}$$

or

$$\sin(\alpha - x + \psi_1) = 0, \tag{91}$$

which vanishes when

$$\alpha - x + \psi_1 = -n\pi\ (n \text{ an integer or } 0); \tag{92}$$

that is, when

$$x = \alpha + n\pi + \psi_1. \tag{93}$$

We may solve this equation in exactly the same way as in III we solved the equation (39); we need only replace each f-function by the corresponding g-function, due regard being paid to the algebraic signs. We thus obtain

$$x = \alpha + n\pi + \frac{g_1}{\alpha + n\pi} - \frac{g_3 - g_1 g_2 + \dfrac{g_1^3}{3} + g_1^2}{(\alpha + n\pi)^3} + \cdots \tag{94}$$

If we express this in terms of the f-functions, we have

$$x = \alpha + n\pi - \frac{f_1 + \frac{1}{2}}{\alpha + n\pi} + \frac{f_3 + 2f_2 - f_1 f_2 - 2f_1^2 - \frac{3}{2}f_1 + \frac{f_1^3}{3} - \frac{7}{24}}{(\alpha + n\pi)^3} + \dots, \tag{95}$$

or, in terms of p and q,

$$x = \alpha + n\pi - \frac{\frac{p+1}{2}}{\alpha + n\pi} - \frac{\frac{5p^2}{24} + \frac{7p}{24} + q}{(\alpha + n\pi)^3}$$
$$+ \frac{-\frac{5}{48}p^4 + \frac{103}{960}p^3 - \frac{169}{240}p^2 - \frac{107}{24}p - \frac{113}{480} + q\left(\frac{23}{20}p + \frac{11}{2}\right)}{(\alpha + n\pi)^5} + \dots \tag{96}$$

If we put for x its value $x = \frac{ke^u}{2}$, we have, finally,

$$e^u = \frac{1}{q^{\frac{1}{4}}}\left[\alpha + n\pi - \frac{\frac{p+1}{2}}{\alpha + n\pi} - \frac{\frac{5}{24}p^2 + \frac{7}{24}p + q}{(\alpha + n\pi)^3} + \dots\right]. \tag{97}$$

This formula gives, then, the position of the $(n+1)$th maxima of the curve U_a. Here also a change in the value of the undetermined constant α has practically the effect of displacing the maxima or minima along the axis. A comparison of the equations (96) and (44) shows further that the maxima and minima as given by (96) alternate with the crossings of the axis as given by (44).

The magnitude of the maxima and minima ordinates, or indeed of any ordinate, is of course given by

$$U_a = \frac{C}{e^{\frac{u}{2}}}\left[P\cos\left(\alpha - \frac{ke^u}{2}\right) + Q\sin\left(\alpha - \frac{ke^u}{2}\right)\right], \tag{20}$$

which may be written in the form

$$U_a = \frac{C}{e^{\frac{u}{2}}} M \cos\left(\alpha - \frac{ke^u}{2} - \psi\right), \tag{98}$$

where M is given by (61) as

$$M = 1 - \frac{p}{q^{\frac{1}{2}}e^{2u}} + \frac{\frac{5}{2}p^2 + 6p - 4q}{qe^{4u}} + \dots \tag{99}$$

and ψ is given by

$$\psi = \tan^{-1}\left(\frac{p}{q^{\frac{1}{4}}e^u} + \frac{\frac{p^3}{3} - \frac{p^2}{3} - 2p + 8q}{q^{\frac{3}{4}}e^{3u}}\right.$$
$$\left. + \frac{\frac{13}{60}p^5 - \frac{5}{12}p^4 - \frac{p^3}{10} - \frac{163}{15}p^2 + \frac{316}{3}p + q\left(8p^2 - \frac{104}{5}p - 96q\right)}{q^{\frac{5}{4}}e^{5u}} + \dots\right) \tag{100}$$

The length of any ordinate is then given by (98), thus depending upon both the undetermined constants C and α. This formula also shows how, for increasing u, the lengths of the maximum and minimum ordinates continually decrease.

V. *Numerical Verification.*

The equation considered by Heine, Dannacher and Butts is written in the form

$$\frac{d^2E}{d\phi^2}+\left(\frac{8}{b}\cos 2\phi + 4z\right)E=0. \tag{101}$$

If we put $\phi = iu$, we have

$$\frac{d^2E}{du^2}-\left(\frac{8}{b}\cosh 2u + 4z\right)E=0. \tag{102}$$

If now we make the following substitutions:

$$\left.\begin{aligned} &\cosh 2u = 2\cosh^2 u - 1,\\ &\frac{16}{b}=k^2,\\ &\mathfrak{B}+\frac{8}{b}=4z, \end{aligned}\right\} \tag{103}$$

equation (102) becomes

$$\frac{d^2E}{du^2}-(k^2\cosh^2 u+\mathfrak{B})E=0, \tag{104}$$

which differs from equation (1) only in the sign of the second term. Now, by means of transformations similar to those employed in I, we reduce the equation (104) to the following form:

$$\frac{d^2E_1}{dx^2}-\left(1-\frac{r}{x^2}-\frac{s}{x^4}\right)E_1=0, \tag{105}$$

where

$$\left.\begin{aligned} r&=-\frac{1}{4}-\frac{k^2}{2}-\mathfrak{B},\\ s&=-\frac{k^4}{16}. \end{aligned}\right\} \tag{106}$$

For large values of x, (105) becomes approximately

$$\frac{d^2E_1}{dx^2}-E_1=0, \tag{107}$$

so that we may assume as an asymptotic solution of (105), for large values of x,

$$E_1 = e^x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \dots\right). \tag{108}$$

Proceeding now as in I, we find, for the semi-convergent solution of (102) for large values of x,

$$E = A_0 \frac{e^{\frac{ke^u}{2}}}{e^{\frac{u}{2}}}\left[1 + \frac{h_1}{\frac{ke^u}{2}} + \frac{h_2}{\left(\frac{ke^u}{2}\right)^2} + \frac{h_3}{\left(\frac{ke^u}{2}\right)^3} + \dots\right], \tag{109}$$

where A_0 is an undetermined constant, and where the $h_1, h_2, h_3, \dots$ are abbreviations for the following:

$$\left.\begin{aligned} h_1 &= \frac{r}{2}, \\ h_2 &= \frac{r(r+2)}{2^2 \,.\, 2\,!}, \\ h_3 &= \frac{r(r+2)(r+6)}{2^3 \,.\, 3\,!} + \frac{s}{2 \,.\, 3}, \\ &\dots\dots\dots\dots\dots, \\ h_n &= \frac{h_{n-1}[r + n(n-1)] + h_{n-3} \,.\, s}{2(n-1)}; \end{aligned}\right\} \tag{110)*$$

and where

$$\left.\begin{aligned} r &= -\frac{1}{4} - \frac{k^2}{2} - \mathfrak{B}, \\ s &= -\frac{k^4}{16}. \end{aligned}\right\} \tag{111)*$$

Also between the constants of equation (102) and (109) there exist the following relations:

$$\left.\begin{aligned} r &= -\frac{1}{4} - 4z, \\ s &= -\frac{16}{b^2}, \\ k &= \frac{4}{\sqrt{b}}. \end{aligned}\right\} \tag{112}$$

* It may be remarked that $h_1, h_2, h_3, \dots$ are the same as $f_1, f_2, f_3, \dots$, as defined by (12), except that here all the signs are positive. Also $p = -r$ and $q = -s$.

A solution of (102) as given by Heine is

$$E(u) = \frac{a_0}{2} + a_1 \cosh 2u + a_2 \cosh 4u + \ldots, \tag{113}$$

where $a_0 = 1$, $a_1 = \frac{1}{2} bz$, and the other a's are given by the recurrence formula

$$a_{n+1} = b(n^2 - z)a_n - a_{n-1}. \tag{114}$$

This series (113) converges (for any assumed value of b) only for values of z which are the roots of a certain transcendental function.* For large values of u it at first diverges rapidly, and ultimately converges slowly. The series given by (109), on the other hand, at first converges rapidly, for reasonably large values of u, and then diverges rapidly. It may be used for approximate numerical calculation, provided care is taken not to include any terms after divergence begins. We wish to show by means of a numerical example how the asymptotic expansion given in (109) leads to the same result as (113), but with much less labor in computation.

It has been shown by Butts† that if b be taken equal to .1, then one root of the *limiting function*, that is, one value of z for which the series (113) converges, is 5.58134 (correct to five decimal places). He gives the values of the coefficients computed for this value of b and z as

$$\left.\begin{aligned} a_0 &= 1.0000, \\ a_1 &= -0.2790, \\ a_2 &= -0.8721, \\ a_3 &= 0.4169, \\ a_4 &= 1.0147, \\ a_5 &= 0.6402, \\ a_6 &= 0.2284, \\ a_7 &= 0.0548, \\ a_8 &= 0.0096, \\ a_9 &= 0.0015, \\ a_{10} &= 0.0001. \end{aligned}\right\} \tag{115‡}$$

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* See Dannacher, *l. c.*, p. 29.

† W. H. Butts., *l. c.*, p 20.

‡ a_9 and a_{10} are not exactly as given by Butts, but have been changed slightly by means of Heine's formula $a_{n+1} < \frac{a_n}{bn^2}$, which holds from a certain n on.

If we put now $u = .7$, equation (113) gives us

$$\begin{aligned} E(.7) = .5 - .2790 \cosh 1.4 - .8721 \cosh 2.8 + .4169 \cosh 4.2 \\ + 1.0147 \cosh 5.6 + .6402 \cosh 7.0 + .2284 \cosh 8.4 \\ + .0547 \cosh 9.8 + .0096 \cosh 11.2 + .0015 \cosh 12.6 \\ + .0001 \cosh 14. \end{aligned} \tag{116}$$

With a four-place logarithm table this gives

$$\begin{aligned} E(.7) = .5000 - .6000 - 7.7980 + 13.9100 + 136.9000 + 351.0000 \\ + 507.3000 + 494.2000 + 354.2000 + 222.5000 + 60.7500 = 2133. \end{aligned} \tag{117}$$

In order to use equation (109) we must in (112) substitute the values of b and z, namely .1 and 5.58134. This gives

$$\left.\begin{aligned} r &= -22.47536, \\ s &= -1600, \\ k &= 4\sqrt{10}. \end{aligned}\right\} \tag{118}$$

Substituting these in (109), putting $u = .7$, and using four-place logarithms as before, we have, using four terms of the series,

$$\begin{aligned} E(.7) = 239500 A_0[1 - .8824 + .3546 - (.1291 + .0765) \\ + (.0079 + .0138 + .0853)] = 239500 A_0(.3727). \end{aligned} \tag{119}$$

A comparison of (117) and (119) serves now to determine A_0. We have, using four-place logarithms as before,

$$A_0 = .02388. \tag{120}$$

This value of A_0 may now be employed in finding the value of $E(u)$ for other arguments, by means of the series (109). For example, if we put $u = .8$ and use, as before, four terms of the series and four-figure logarithms, we have

$$\begin{aligned} E(.8) = 869600 A_0[1 - .7976 + .2904 - (.0956 + .0566) \\ + (.0052 + .0089 + 0572)] = 869600(.02388)(.4119) = 8552. \end{aligned} \tag{121}$$

We may verify this by means of (113). Carrying the series to ten terms and using, as before, four-place logarithms, we have

$$\begin{aligned} E(.8) = .5000 - .8672 - 8.5130 + 25.3400 + 305.20 + 954.2 \\ + 1766 + 2004 + 1599 + 1345 + 444.0 = 8434. \end{aligned} \tag{122}$$

Considering the roughness of the approximation used, and particularly the possible incorrectness of a_6, a_7, a_8, a_9, a_{10}, this may be regarded as a very close

agreement. The fact that (122) is too small may be accounted for by the fact that the next few terms would contribute materially to the result. In using series (113) for large values of the argument it would be necessary to know the value of z and the corresponding values of $a_1, a_2, \ldots$ with much greater exactness. This fact determined in a way the choice of the particular values $u = .7$ and $u = .8$ in this illustration. For values of u much smaller than .7, (109) does not converge at all; and for values of u much greater than .8, the use of (113) is too uncertain. That the example is a particularly unfavorable one is due to the fact that b was chosen so small and z large in comparison, which has the effect of making both r and s large. Under more favorable circumstances (109) would converge much more rapidly, and for smaller values of the argument.*

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*Since the above was written, my attention has been called to an article by R. C. Maclaurin in Vol. XVII, Pt. I, of the *Transactions of the Cambridge Philosophical Society*. In this article, which is entitled "On the Solutions of the Equation $(\Delta^2 + k^2)\psi = 0$ in Elliptic Coordinates and their Physical Applications," Mr. Maclaurin reduces the equation $(\Delta^2 + k^2)\psi = 0$ to the form

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - p^2)y = 0.$$

He then obtains power-series solutions in the neighborhood of -1, $+1$, ∞, this latter being asymptotic. He suggests the possibility of using this asymptotic expansion in finding expressions for the roots of the function itself and of its first derivative, and calls attention to the importance of these roots in practical applications.